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The Search for a Realistic Superstring Vacuum [and Discussion]

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The search for a realistic superstring vacuum

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Recent progress in understanding heterotic string compactification utilizes abstract algebraic methods. In particular, Gepner has given a prescription for constructing exactly soluble Calabi–Yau compactifications by using $N = 2$ minimal models. A large class of new $N = 2$ models, discovered by Kazama & Suzuki, can also be used. Recent progress in understanding $N = 2$ models and Calabi–Yau spaces by using mathematical techniques of singularity theory improves the prospects of a complete classification. A key question is whether a classical solution can give a reasonable first approximation to the exact quantum ground state even though string theory is strongly coupled and the perturbation expansion diverges.

Introduction

In the four years that have passed since string theory emerged as a popular approach to unification we have learned a great deal about many facets of the programme (Green *et al.* 1987). Two general categories of issues have emerged as critical and received much scrutiny in recent times. One is the quest for a genuine non-perturbative formulation of quantum string theory. A very broad range of ideas and approaches have been put forward, but as yet none has emerged as the consensus favourite. The second category of questions, which is the subject of this review, concerns the search for classical solutions of the heterotic string theory, with particular emphasis on those that are most promising for achieving phenomenological success.

Four years ago string theory was touted for its uniqueness. In a certain sense that is still a viable point of view today. There are only three theories – type I, type II, and heterotic – that appear to be internally consistent when analysed in perturbation theory. Each is completely free of parameters or other arbitrariness. What has become clear in the intervening years is that this uniqueness at the level of the fundamental equations (whatever they are) is not reflected at the level of classical solutions. A bewildering proliferation of classical solutions have been discovered by a variety of techniques. However, there is a unifying principle. Four dimensions can be taken to be flat Minkowski space with the remaining degrees of freedom described by an arbitrary conformal field theory having suitable central charges, supersymmetry and modular invariance. All classical solutions that have been proposed, whether or not expressed in these terms, can be interpreted as particular constructions of the internal conformal field theory.

Unfortunately, at the level of perturbative analysis, this programme has some disturbing arbitrariness. There is no compelling theoretical reason to separate off four-dimensional space-time or to require that it be a Minkowski space. Moreover, the possibilities for the internal conformal field theories are very numerous. The number can be reduced by imposing some physical requirements such as $N = 1$ supersymmetry (to solve the hierarchy problem) or a particular number of families.

One of the first proposals, Calabi–Yau compactification (Candelas *et al.* 1985), still looks the most promising. From the abstract conformal field theory point of view it corresponds to a class of (2, 2) superconformal models with central charge $c = 9$. The abstract approach treats related orbifold compactifications as part of the same category. The detailed procedure for turning an arbitrary (2, 2) superconformal model with $c = 9$ into an $N = 1, D = 4$ heterotic string solution with E_6 families of quarks and leptons has been worked out by Gepner (1987*b*).

Most of this review is concerned with the description of various techniques that have been developed for constructing (2, 2) superconformal models. Many examples, including the minimal models considered by Gepner, can be constructed from Kac–Moody algebra cosets, by using the method of Goddard *et al.* (1985). A supersymmetric extension of this method has been formulated by Kazama & Suzuki (1988) and applied to the construction of many new $N = 2$ superconformal models, which can also be used in the construction of the internal $c = 9$ model.

Gepner gave convincing evidence that certain of his models correspond to known Calabi–Yau compactifications. This seemed miraculous at the time, because his construction is entirely algebraic and uses exactly soluble minimal models. Calabi–Yau spaces on the other hand are very complicated geometric structures, none of which has a known metric. The origins of this 'miracle' are much better understood now in view of recent developments applying the mathematical techniques of singularity theory to the description of $N = 2$ models.

Conformal field theory

This section briefly reviews some of the central ideas in conformal field theory that are required in the sequel. For more thorough and systematic discussions the reader is referred to the review articles (Peskin 1987; Banks 1987; Ginsparg 1988).

It is extremely convenient to make a Wick rotation $\tau \rightarrow i\tau$ so as to euclideanize the string world-sheet and thereby make the metric $h_{\alpha\beta}$ positive definite. Having done this, we can introduce complex coordinates (in local patches)

$$z = e^{\tau + i\sigma} \quad \text{and} \quad \bar{z} = e^{\tau - i\sigma} \tag{1}$$

and regard the world sheet as a Riemann surface. The gauge invariances $\tau \pm \sigma \rightarrow f_{\pm}(\tau \pm \sigma)$ become conformal mappings $z \rightarrow \tilde{z}(z)$ and similarly for $\bar{z}$. Thus we are led to consider conformally invariant two-dimensional field theory (Belavin *et al.* 1984). These transformations are generated by the energy–momentum tensor components $T(z)$ and $\bar{T}(\bar{z})$.

For a conformal dimension h field we write

$$\phi(z) = \sum_{-\infty}^{\infty} \frac{\phi_n}{z^{n+h}}. \tag{2}$$

More generally, conformal fields are functions of z and $\bar{z}$ and thus have a pair of dimensions $(h, \bar{h})$. However, to simplify writing I will usually suppress $\bar{z}$ dependences. If the ghosts are included (so that the conformal anomaly cancels), $T(z)$ has dimension (2, 0) and $\bar{T}(\bar{z})$ has dimension (0, 2).

Under a finite transformation $z \rightarrow \tilde{z}(z)$, a conformal dimension h field transforms as follows

$$\phi(z) \rightarrow (\partial\tilde{z}/\partial z)^h \tilde{\phi}(\tilde{z}). \tag{3}$$

One sometimes says that ϕ is an 'h-form', because $\phi(z)\,(\mathrm{d}z)^h$ is invariant. The infinitesimal transformation is determined by the operator product expansion (OPE)

$$T(z)\,\phi(w) \sim h\phi(w)/(z-w)^2 + \partial\phi(w)/(z-w) + \ldots. \tag{4}$$

The symbol ∂ means differentiation with respect to w, of course. The dots represent non-singular terms. The $N = 1$ superconformal algebra corresponds to the OPES

$$\left.\begin{aligned} T(z)\,T(w) &\sim \frac{c}{2(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial T(w)}{z-w} + \ldots, \\ T(z)\,G(w) &\sim \frac{3}{2}\frac{G(w)}{(z-w)^2} + \frac{\partial G(w)}{z-w} + \ldots, \\ G(z)\,G(w) &\sim \frac{2c}{3(z-w)^3} + \frac{2T(w)}{z-w} + \ldots. \end{aligned}\right\} \tag{5}$$

Thus $G(z)$ has dimension $(\frac{3}{2}, 0)$.

Particularly interesting examples of conformal fields are the two-dimensional currents associated with a Lie group symmetry in a conformal field theory (Goddard & Olive 1986). Using current conservation one can show that there is a holomorphic component $J^A(z)$ and an antiholomorphic component $\bar{J}^A(\bar{z})$, just as for T and G. As above, we consider $J^A(z)$ only. The zero modes J_0^A are the generators of a Lie algebra G with

$$[J_0^A, J_0^B] = \mathrm{i}f_{ABC}\,J_0^C. \tag{6}$$

The algebra of the currents $J^A(z)$ is an infinite-dimensional extension of this, known as an affine Lie algebra or as a Kac–Moody algebra $\hat{G}$. These currents have conformal dimension $h = 1$. The Kac–Moody algebra is given by the OPES

$$J^A(z)\,J^B(w) \sim \frac{k\delta^{AB}}{2(z-w)^2} + \frac{\mathrm{i}f_{ABC}\,J^C(w)}{z-w} + \ldots. \tag{7}$$

The parameter k in the Kac–Moody algebra, called the 'level', is analogous to the parameter c in the conformal algebra. For a U(1) Kac–Moody algebra, $\hat{\mathrm{U}}(1)$, it can be absorbed in the normalization of the current. However, for a non-abelian group G, it has an absolute meaning once the normalizations are carefully specified. Rather than giving general formulas, let me simply state that I choose $f_{ABC} = \epsilon_{ABC}$ in the case of SU(2). With this normalization convention, the algebra admits unitary representations if and only if k is a positive integer. Unitarity is essential for us, because we will use such algebras to define the positive-definite Hilbert space of physical states. An important formula, due to Sugawara (1986) and Sommerfield (1986), gives the energy–momentum tensor associated with an arbitrary Kac–Moody algebra:

$$T(z) = \frac{1}{k + \tilde{h}_G} \sum_{A=1}^{\dim G} :J^A(z)\,J^A(z):. \tag{8}$$

In the case of simply laced algebras ($G = A, D, E$) the dual Coxeter number $\tilde{h}_G$ equals c_A, where c_A is the quadratic Casimir number of the adjoint representation defined (with our normalization conventions) by

$$f_{ABC}\,f_{A'BC} = c_A\,\delta_{AA'}. \tag{9}$$

The associated central charge is $$c = k \dim G/(k + \tilde{h}_G). \tag{10}$$

For example, in the case of $\widehat{\mathrm{SU}(2)}_k$, $\tilde{h} = 2$ and $c = 3k/(k+2)$.

An important restriction on the operator content of a conformal field theory is provided by the requirement of modular invariance (Cardy 1986). Specifically, it is necessary that the partition function

$$Z(\tau) = \sum N_{h_i \bar{h}_i} \chi_{h_i}(\tau)\, \chi^*_{\bar{h}_i}(\tau) \tag{11}$$

be invariant under the transformation group

$$\tau \rightarrow (a\tau + b)/(c\tau + d), \tag{12}$$

where a, b, c, d are integers satisfying $ad - bc = 1$. This group is generated by the transformations $T: \tau \rightarrow \tau + 1$ and $S: \tau \rightarrow -1/\tau$. The coefficients $N_{h_i \bar{h}_i}$ in (11) are non-negative integers representing the multiplicities of operators with conformal dimensions $(h_i, \bar{h}_i)$. The conformal characters $\chi_{h_i}(\tau)$ are the trace of $e^{2\pi i \tau (L_0 - \frac{1}{24}c)}$ in the module defined by the highest weight state $|h_i\rangle$ and its descendants. Clearly T invariance requires that the spins $h_i - \bar{h}_i$ be integers. Given a chiral algebra, such as the Kac–Moody algebras, it is in general a difficult problem to find all the possible modular invariant partition functions and to classify the associated conformal field theories. However, in the case of the Kac–Moody algebras $\widehat{\mathrm{SU}}(2)_k$ the problem has been completely solved.

From the representation theory of $\widehat{\mathrm{SU}}(2)_k$ one knows that the possible dimensions of primary fields are $l = 0, \frac{1}{2}, \ldots, \frac{1}{2}k$. Letting $\lambda = 2l + 1$ and $N = 2(k+2)$, the characters are (Kac & Peterson 1984)

$$\chi_\lambda(\tau) = \frac{1}{\eta^3(\tau)} \sum_{n=-\infty}^{\infty} (nN + \lambda) \exp\,[\mathrm{i}\pi\tau(nN+\lambda)^2/N], \tag{13}$$

where $\eta(\tau)$ is the Dedekind eta function

$$\eta(\tau) = \exp\,(\tfrac{1}{12}\pi \mathrm{i}\tau) \sum_{n=1}^{\infty} [-\exp\,(2\pi \mathrm{i} n\tau)]. \tag{14}$$

Under an S transformation one has

$$\chi_\lambda(-1/\tau) = \sqrt{\left(\frac{2}{k+2}\right)} \sum_{\lambda'=1}^{k+1} \sin\left(\frac{\pi\lambda\lambda'}{k+2}\right) \chi_{\lambda'}(\tau).$$

By using the facts given above, the following theorem has been proved (Capelli *et al.* 1987; Gepner & Qiu 1987*a*): modular-invariant $\widehat{\mathrm{SU}}(2)$ partition functions are in one-to-one correspondence with simply laced Lie algebras. In each case $k+2$ is the Coxeter number of the corresponding algebra. Moreover, the multiplicities of the diagonal terms are the Betti numbers of the corresponding algebras. The partition functions are listed in table 1. Because $\widehat{\mathrm{SU}}(2)$ enters in a crucial way in the construction of various other conformal field theories, this result is directly relevant for them, as well.

TABLE 1. MODULAR-INVARIANT PARTITION FUNCTIONS FOR $\mathrm{SU}(2)_k$ (Capelli *et al.* 1987.)

A_{k+1}	$\sum_{\lambda=1}^{k+1} \lvert\chi_\lambda\rvert^2$	$k \geqslant 1$
$D_{2\rho+2}$	$\sum_{\lambda_{\mathrm{odd}}=1}^{2\rho-1} \lvert\chi_\lambda + \chi_{4\rho+2-\lambda}\rvert^2 + 2\lvert\chi_{2\rho+1}\rvert^2$	$k = 4\rho,\ \rho \geqslant 1$
$D_{2\rho+1}$	$\sum_{\lambda_{\mathrm{odd}}=1}^{4\rho-1} \lvert\chi_\lambda\rvert^2 + \lvert\chi_{2\rho}\rvert^2 + \sum_{\lambda_{\mathrm{even}}=2}^{2\rho-2} (\chi_\lambda \chi^*_{4\rho-\lambda} + \mathrm{c.c.})$	$k = 4\rho-2,\ \rho \geqslant 2$
E_6	$\lvert\chi_1+\chi_7\rvert^2 + \lvert\chi_4+\chi_8\rvert^2 + \lvert\chi_5+\chi_{11}\rvert^2$	$k = 10$
E_7	$\lvert\chi_1+\chi_{17}\rvert^2 + \lvert\chi_5+\chi_{13}\rvert^2 + \lvert\chi_7+\chi_{11}\rvert^2 + \lvert\chi_9\rvert^2 + [(\chi_3+\chi_{15})\chi_9^* + \mathrm{c.c.}]$	$k = 16$
E_8	$\lvert\chi_1+\chi_{11}\rvert^2 + \lvert\chi_{19}+\chi_{29}\rvert^2 + \lvert\chi_7+\chi_{13}+\chi_{17}+\chi_{23}\rvert^2$	$k = 28$

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An important first step towards finding the general result for arbitrary Kac–Moody algebras has been taken by Gepner (1987*a*). He gives a complete list of modular-invariant partition functions without the restriction $N_{h_i \bar{h}_i} \geqslant 0$. In the form given, it is a very non-trivial problem to determine the subset that satisfies this restriction.

$N = 2$ SUPERCONFORMAL SYMMETRY

By definition, an N-extended superconformal symmetry algebra is one containing N dimension-$\frac{3}{2}$ fermionic generators $G^\alpha(z)$; $\alpha = 1, 2, \ldots, N$. It is also required that the OPE $G^\alpha(z)\, G^\beta(w)$ contain the term $2T(w)\,\delta^{\alpha\beta}/(z-w)$, where $T(w)$ is energy–momentum tensor. The $N=2$ algebra, in addition to T, G^1, G^2, also contains a dimension-one current $J(z)$; J can be regarded as defining an abelian Kac–Moody algebra $\hat{U}(1)$. The $N = 2$ superconformal algebra is given by

$$\left.\begin{aligned} T(z)\,T(w) &\sim \frac{c}{2(z-w)^4}+\frac{2T(w)}{(z-w)^2}+\frac{\partial T(w)}{z-w}+\ldots,\\ G^\alpha(z)\,G^\beta(w) &\sim \left(\frac{2c}{3(z-w)^3}+\frac{2T(w)}{z-w}\right)\delta^{\alpha\beta}+\mathrm{i}\left(\frac{2J(w)}{(z-w)^2}+\frac{\partial J(w)}{z-w}\right)\epsilon^{\alpha\beta}+\ldots,\\ T(z)\,G^\alpha(w) &\sim \frac{3}{2}\frac{G^\alpha(w)}{(z-w)^2}+\frac{\partial G^\alpha(w)}{z-w}+\ldots,\\ T(z)\,J(w) &\sim \frac{J(w)}{(z-w)^2}+\frac{\partial J(w)}{z-w}+\ldots,\\ J(z)\,G^\alpha(w) &\sim \mathrm{i}\epsilon^{\alpha\beta}\frac{G^\beta(w)}{z-w}+\ldots,\\ J(z)\,J(w) &\sim \frac{1}{3}\frac{c}{(z-w)^2}+\ldots. \end{aligned}\right\} \tag{15}$$

The combinations $G^\pm(z) = (G^1(z) \pm \mathrm{i}G^2(z))/\sqrt{2}$ are sometimes convenient to consider, as $G^+(z)\,G^+(w)$ and $G^-(z)\,G^-(w)$ are non-singular, whereas

$$G^+(z)\,G^-(w) \sim \frac{2c}{3(z-w)^3}+\frac{2J(w)}{(z-w)^2}+\frac{2T(w)+\partial J(w)}{z-w}+\ldots. \tag{16}$$

In the case of the $N=1$ superconformal algebra we distinguish two sectors, NS and R, according to whether $G(z)$ has half-integral or integral modes. In the case of the $N=2$ algebra there are three distinct sectors that can be distinguished (Boucher *et al.* 1986):

(1) the R sector in which all operators have integer modes;

(2) the NS sector in which $G^1(z)$ and $G^2(z)$ have half-integral modes and $T(z)$ and $J(z)$ have integral modes;

(3) the T sector in which $G^1(z)$ and $J(z)$ have half-integral modes and $G^2(z)$ and $T(z)$ have integral modes. The T sector does not seem to play a role in applications to string theory.

When $J(z)$ has integer modes, namely in the R and NS sectors, it has a zero mode J_0, which is a U(1) charge. In these sectors, we can classify highest-weight states (and hence primary fields) not only by the eigenvalue h of the operator L_0, but also by the eigenvalue q of J_0. In the twisted sector $J(z)$ has no zero mode and hence there is no such charge to be defined.

To explicitly construct a large class of $N=2$ models it is convenient to consider super-

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Kac–Moody algebras. They contain dimension-$\frac{1}{2}$ fermionic currents $j^A(z)$ as superpartners of the usual dimension-1 bosonic currents $J^A(z)$. Because the $j^A(z)$ belong to the adjoint representation,

$$J^A(z)\,j^B(w) \sim \frac{\mathrm{i}f_{ABC}\,j^C(w)}{z-w}+\dots. \tag{17}$$

They have free fermion commutation relations, so choosing a convenient normalization,

$$j^A(z)\,j^B(w) \sim \frac{k\delta^{AB}}{2(z-w)}+\dots. \tag{18}$$

The unitary representation of a super-Kac–Moody algebra with the lowest possible level is given by setting the bosonic currents $J^A(z)$ equal to the fermion bilinears

$$J_f^A(z) = -(\mathrm{i}/k)\,f^{ABC}j^B(z)\,j^C(z). \tag{19}$$

In this case it is easy to see that the conformal anomaly is just that of $\dim G$ free fermions $c_f = \dim\frac{1}{2}G$, and the level of the Kac–Moody algebra is $k_f = c_A(G)$.

By using this special representation, the general representation of a super-Kac–Moody algebra can be obtained (Kac & Todorov 1985). Letting

$$J^A(z) = J_f^A(z)+\hat{J}^A(z), \tag{20}$$

one sees that $\hat{J}^A(z)$ defines a Kac–Moody algebra that is independent of the fermion fields. In other words, $\hat{J}^A(z)\,j^B(w)$ and $\hat{J}^A(z)\,J_f^B(w)$ are non-singular. If we therefore consider a level $\hat{k}$ representation of the algebra given by the $\hat{J}^A(z)$, we obtain a representation of the $J^A(z)$ algebra with level

$$k = \hat{k}+c_A(G) = \hat{k}+\tilde{h}_G \tag{21}$$

and central charge

$$c_G = \hat{k}\dim G/(\hat{k}+\tilde{h}_G)+\tfrac{1}{2}\dim G. \tag{22}$$

The allowed values of $\hat{k}$ are $0, 1, 2, \dots$, where the choice $\hat{k} = 0$ implies setting $\hat{J}^A = 0$. For simplicity it is assumed that G is simply laced so that $c_A(G) = \tilde{h}_G$, where $c_A(G)$ is the Casimir number defined in (9). The corresponding energy–momentum tensor associated with the group G is

$$T_G(z) = (1/k)\,(:\hat{J}^A(z)\,\hat{J}^A(z):-:j^A(z)\,\partial j^A(z):), \tag{23}$$

and the supercurrent is

$$G_G(z) = (2/k)\,(j^A(z)\,\hat{J}^A(z)-\tfrac{1}{3}(\mathrm{i}/k)f_{ABC}\,j^A(z)\,j^B(z)\,j^C(z)). \tag{24}$$

It is now easy to generalize the GKO construction (Goddard *et al.* 1985) to super-Kac–Moody algebras (Kazama & Suzuki 1988*a*). Let $J^a(z), a = 1, \dots, \dim H$, denote the generators of $\hat{H}$ as before. Then, just as for $\hat{G}$, we can write

$$J^a(z) = \tilde{J}^a(z)-(\mathrm{i}/k)f_{abc}\,j^b(z)\,j^c(z), \tag{25}$$

where f_{abc} are the H structure constants and $j^a(z)$ are $\dim H$ of the free fermions. The remaining $\dim G-\dim H$ free fermions are denoted $j^{\bar{a}}(z)$. The H supercurrent is

$$G_H(z) = (2/k)\,(j^a(z)\,\tilde{J}^a(z)-\tfrac{1}{3}(\mathrm{i}/k)f_{abc}\,j^a(z)\,j^b(z)\,j^c(z)), \tag{26}$$

and the coset model is defined by the difference

$$G(z) = G_G(z)-G_H(z) = (2/k)\,(j^{\bar{a}}(z)\,\hat{J}^{\bar{a}}(z)-\tfrac{1}{3}(\mathrm{i}/k)f_{\bar{a}\bar{b}\bar{c}}\,j^{\bar{a}}(z)\,j^{\bar{b}}(z)\,j^{\bar{c}}(z)). \tag{27}$$

Just as in the ordinary GKO construction, one can easily show that this has non-singular OPEs with the H currents $\hat{J}^a(w)$ and $j^a(w)$. It is therefore guaranteed to define an $N=1$ superconformal theory with central charge $c = c_G - c_H$, where

$$c_G = \tfrac{3}{2}\dim G - (\tilde{h}_G/k)\dim G \tag{28}$$

and similarly for c_H. The corresponding energy–momentum tensor is

$$T = \frac{1}{k}\left(\hat{J}^{\bar a}\hat{J}^{\bar a} - \frac{\hat k}{k} j^{\bar a}\,\partial j^{\bar a} + \frac{2\mathrm{i}}{k} f_{a\bar b\bar c}\,\hat{J}^a j^{\bar b} j^{\bar c} - \frac{1}{k} f_{\bar a\bar p\bar q}\, f_{\bar b\bar p\bar q}\, j^{\bar a}\,\partial j^{\bar b} - \frac{1}{k^2} f_{\bar a\bar b\bar c}\, f_{\bar a\bar d\bar e}\, j^{\bar b} j^{\bar c} j^{\bar d} j^{\bar e}\right). \tag{29}$$

Let us examine the SU(2) case in more detail. In this case dim $(G/H) = 2$ and $k = \hat{k}+2$, so that

$$c = 3\hat{k}/(\hat{k}+2), \tag{30}$$

providing a derivation of the $N=2$ minimal models. The index $\bar a$ now takes the values 1 and 2, whereas the index a takes the value 3. Thus the equations for $T(z)$ and $G(z)$ become

$$T = \frac{1}{k}[(\hat{J}^1)^2 + (\hat{J}^2)^2] + \frac{4\mathrm{i}}{k^2}\hat{J}^3 j^1 j^2 - \frac{\hat k}{k^2}(j^1\,\partial j^1 + j^2\,\partial j^2) \tag{31}$$

and

$$G \equiv G^1 = (2/k)\,(j^1\hat{J}^1 + j^2\hat{J}^2). \tag{32}$$

The construction of the $N=2$ representation is completed by the identifications

$$G^2(z) = (2/k)\,(j^1\hat{J}^2 - j^2\hat{J}^1) \tag{33}$$

and

$$J = (2/k)\,\hat{J}^3 + (2\mathrm{i}\hat{k}/k^2)\, j^1 j^2. \tag{34}$$

As a final check we can compute

$$J(z)\,J(w) \approx \tfrac{1}{3}c/(z-w)^2, \tag{35}$$

with $c = 3\hat{k}/(\hat{k}+2)$, as required. Not surprisingly, the modular-invariant partition functions of the minimal models have the same ADE classification that we presented for $\widehat{\mathrm{SU}}(2)$ models.

We have seen that it is possible to associate unitary representations of the $N=1$ superconformal algebra with cosets by a generalized GKO construction. We saw that the theory admits a second supercurrent and thus furnishes an $N=2$ model in the case of SU(2)/U(1), corresponding to the $N=2$ minimal models. A natural question is, What is the most general choice of G/H for which the $N=1$ construction in fact defines an $N=2$ algebra? This question has been examined by Kazama & Suzuki (1988). They find that there is a second supercurrent of the form

$$G^2 = (2/k)\,(h_{\bar a\bar b}\, j^{\bar a}\hat{J}^{\bar b} - \tfrac{1}{3}(\mathrm{i}/k)\, S_{\bar a\bar b\bar c}\, j^{\bar a} j^{\bar b} j^{\bar c}) \tag{36}$$

provided that the following conditions are satisfied:

(i) $h_{\bar a\bar b} = -h_{\bar b\bar a}$ and $h_{\bar a\bar p}h_{\bar p\bar b} = -\delta_{\bar a\bar b}$,
(ii) $h_{\bar a\bar p}f_{\bar p\bar b e} = h_{\bar b\bar p}f_{\bar p\bar a e}$,
(iii) $f_{\bar a\bar b\bar c} = h_{\bar a\bar p}h_{\bar b\bar q}f_{\bar p\bar q\bar c} + 2$ perms,
(iv) $S_{\bar a\bar b\bar c} = h_{\bar a\bar p}h_{\bar b\bar q}h_{\bar c\bar r}f_{\bar p\bar q\bar r}$.

Note that (i) implies that $h_{\bar a\bar b}$ is a complex structure for the coset manifold G/H.

To analyse the implications of these equations, let us define $\phi_e = h_{\bar a\bar b}f_{\bar a\bar b e}$ and consider $X_{cd} =$

$f_{cde}\phi_e$. As $f_{cd\bar{e}} = 0$, by the H group property, $X_{cd} = h_{\bar{a}\bar{b}} f_{\bar{a}\bar{b}E} f_{cdE}$. The Jacobi identity allows us to cycle the indices $\bar{a}, \bar{b}, c$ on the fs. Also, using the antisymmetry of $h_{\bar{a}\bar{b}}$, one obtains

$$X_{cd} = -2h_{\bar{a}\bar{b}} f_{c\bar{a}E} f_{\bar{b}dE} = -2h_{\bar{a}\bar{b}} f_{c\bar{a}\bar{e}} f_{\bar{b}d\bar{e}} = -2h_{\bar{a}\bar{e}} f_{c\bar{a}\bar{b}} f_{\bar{b}d\bar{e}}, \tag{37}$$

where the last step uses property (ii). Now interchanging the labels $\bar{b}$ and $\bar{e}$ we see that X_{cd} is equal to its negative and hence vanishes. It therefore follows that ϕ_e can only be non-zero if e corresponds to a U(1) factor in H. Thus we conclude that a necessary condition for $N = 2$ superconformal symmetry is that H contain a U(1) factor.

Let us now consider the special case of a symmetric space ($f_{\bar{a}\bar{b}\bar{c}} = 0$). Manipulations similar to those of the previous paragraph allow one to show that

$$X_{\bar{c}\bar{d}} = f_{\bar{c}\bar{d}e} \phi_e = c_A h_{\bar{c}\bar{d}}. \tag{38}$$

For hermitian symmetric spaces there is just one U(1) factor. Let $e = 0$ correspond to this U(1) factor, so that the only non-zero component of ϕ is ϕ_0. Then we see that $h_{\bar{a}\bar{b}} \propto f_{\bar{a}\bar{b}0}$, with a normalization determined by condition (i). Thus we are able to associate an $N = 2$ superconformal model to every hermitian symmetric space. These were classified by Cartan and are listed in the book of Helgason (1978). Table 2 lists the irreducible hermitian symmetric spaces (for compact G) and the associated $N = 2$ central charges. There are several cases that give the special value $c = 9$ irreducibly.

Table 2. Hermitian symmetric spaces and the associated $N = 2$ central charges (Kazama & Suzuki 1988.)

G/H	$c_{G/H}$
$\mathrm{SU}(m+n)/\mathrm{SU}(m)\times\mathrm{SU}(n)\times\mathrm{U}(1)$	$3\hat{k}mn/(\hat{k}+m+n)$
$\mathrm{SO}(n+2)/\mathrm{SO}(n)\times\mathrm{SO}(2)$	$3\hat{k}n/(\hat{k}+n)\ n \geqslant 2$
for $n = 1$, $\mathrm{SO}(3)/\mathrm{SO}(2)$	$3\hat{k}/(\hat{k}+2)$
$\mathrm{SO}(2n)/\mathrm{SU}(n)\times\mathrm{U}(1)$	$\frac{3}{2}\hat{k}n(n-1)/(\hat{k}+2n-2)$
$\mathrm{S}p(n)/\mathrm{SU}(n)\times\mathrm{U}(1)$	$\frac{3}{2}\hat{k}(n+1)/(\hat{k}+n+1)$
$\mathrm{E}^6/\mathrm{SO}(10)\times\mathrm{U}(1)$	$48\hat{k}/(\hat{k}+12)$
$\mathrm{E}_7/\mathrm{E}_6\times\mathrm{U}(1)$	$81\hat{k}/(\hat{k}+18)$

(2, 2) compactification

Gepner (1987*c*, 1988) has investigated superstring compactifications that give models with $(10-2n)$-dimensional Poincaré symmetry by describing internal degrees of freedom as a sum of $N = 2$ minimal models with

$$c = \sum_i \frac{3k_i}{k_i+2} = 3n. \tag{39}$$

These constructions can be carried out both for type II superstrings and for heterotic strings. In the latter case modular invariance of partition functions (and hence of loop amplitudes) can be implemented by formulating an analogue of embedding the spin connection in the gauge group. This involves using the same sum of minimal models for the left-movers. In the case of four dimensions ($n = 3$), the remaining $22-9 = 13$ units of c_L are contributed by the level-one Kac–Moody algebra $\mathrm{SO}(10) \otimes \hat{\mathrm{E}}_8$. (At level one the central charge is equal to the rank of the algebra.) The $\widehat{\mathrm{SO}}(10)$ symmetry actually becomes enlarged to E_6, so that one has $\mathrm{E}_6 \times \mathrm{E}_8$ gauge symmetry, as in the case of Calabi–Yau compactification. In general, there is some additional 'accidental' gauge symmetry as well.

Gepner has explored two examples in considerable detail. The first example is based on five copies of the $k = 3$ model. (Note that each one has $c = \frac{9}{5}$.) The second uses one copy of the $k = 1$ model and three copies of the $k = 16_E$ model $(1+3\cdot\frac{8}{3} = 9)$. These are referred to succinctly as 3^5 and $1\cdot 16_E^3$. The subscript E means that the E_7 exceptional affine invariant is used in the construction. He then gives overwhelming circumstantial evidence that these two models correspond to known Calabi–Yau spaces. The evidence includes a count of the numbers of generations and antigenerations as well as an analysis of the discrete symmetries.

One remarkable feature of Gepner's results is that the minimal models are exactly soluble, whereas Calabi–Yau spaces are very unwieldy in general. The construction is completely algebraic, and so it is not understood how geometric structure arises. If one could write a formula for the metric tensor or curvature tensor of the manifold in terms of the conformal field theory, one would have solved a mathematical problem that is usually assumed to be hopeless. Of course, this may not be possible. (Some insight into these matters will be provided later.) The models have extra U(1) gauge symmetries beyond the expected $E_6 \otimes E_8$. There is one associated with each contributing minimal model, but one gets used up in extending SO(10) to E_6. Thus the number of U(1) factors is one less than the number of contributing minimal models (four for 3^5 and three for $1\cdot 16_E^3$). In other examples the gauge symmetry can be extended even further.

Thus the minimal model constructions correspond to Calabi–Yau compactifications at special points of their moduli space where there is a large discrete symmetry group and enhanced gauge symmetry. One wonders whether this makes them too special to be of much interest, or whether these special features could make them physically preferred. Although it is not yet known how to calculate such things, it seems conceivable that non-perturbative effects could induce a potential that depends on the moduli in such a way that the theory would 'roll' to such special points. This would make them particularly good candidates for phenomenology. But even if this is not so, their study still seems to be a useful exercise, because they have so many realistic features. Also, many of these features only depend on the topology of the space and not on the particular choice of moduli. Of course, one is not restricted to minimal models only. If one were to use one of the Kazama–Suzuki models that gives $c = 9$ irreducibly, there would not be any extra 'accidental' gauge symmetry.

It is an interesting challenge to try to make a complete classification of (2, 2) superconformal models with $c = 9$. This is a formidable task, although it does not look as hopeless now, as it did a few years ago. The first step is to examine what can be done with minimal models only. The equation

$$c = \sum_i \frac{3k_i}{k_i+2} = 9 \tag{40}$$

has 168 solutions, which have been enumerated (Lynker & Schimmrigk 1988). However, there is additional freedom in choosing modular invariants of the contributing minimal models. Allowing arbitrary combinations of A, D, and E invariants gives 1176 possibilities. The 228 that only use A and E invariants have been tabulated and the number of generations and antigenerations of E_6 multiplets has been given in each case (Lütken & Ross 1988). There is some redundancy, however, because the E_6 and E_8 minimal models are reducible, as explained later. Many more models could also be formed by using the new $N = 2$ models and by modding out by discrete symmetries. Most cases correspond to Calabi–Yau spaces, but some are orbifolds (Eguchi *et al.* 1988). (There may even be examples with both interpretations!)

Recently, there has been progress in evaluating the Yukawa couplings in these models (Distler & Greene 1988). The $(\mathbf{27})^3$ coupling turns out to be given exactly by the lowest-order (large radius) field theory approximation (Candelas *et al.* 1985), whereas the $(\overline{\mathbf{27}})^3$ is not. The knowledge of these couplings should make it possible to evaluate mass ratios and other quantities of physical interest. Then we can examine how close models of this type can come to agreeing with experiment.

Singularity theory classification of $N=2$ superconformal models

New insights into the classification of (2,2) superconformal models have been obtained recently (Vafa & Warner 1988; Martinec 1988) by using mathematical methods of singularity theory (Arnold 1981) (also known as catastrophe theory). In particular, this work explains how to construct the Calabi–Yau spaces corresponding to all Gepner-type compactifications (Greene *et al.* 1988). This subject is new and developing fast. We will settle here for a brief description of some of the basic concepts.

Many interesting $N=2, d=2$ theories can be described by an action of the form

$$S = \int d^2x\, d^4\theta K(\Phi_i, \bar{\Phi}_i) + \left\{\int d^2x\, d^2\theta W(\Phi_i) + \text{c.c.}\right\}, \tag{41}$$

In this expression the fields Φ_i are chiral $N=2$ superfields, meaning that they are annihilated by certain supercovariant derivatives. In terms of physical states $|\Phi_i\rangle = \Phi_i|0\rangle$, this means that the NS sector descendants $G^a_{-\frac{1}{2}}|\Phi_i\rangle$ are null. The superpotential W is a holomorphic function of the Φ_i. Because of the presence of the 'F term' (the one containing W), these systems can be regarded as $N=2$ Landau–Ginsburg models.

The main idea is to study the renormalization group flow under scale transformations of the two-dimensional metric $g \rightarrow \lambda^2 g$ as $\lambda \rightarrow \infty$. The kinetic term (known as the 'D term') contains only irrelevant operators, and thus W determines the fixed point of the RG flow. Specifically, at a fixed point the chiral fields scale according to

$$\left.\begin{aligned} \Phi_i &\rightarrow \lambda^{\omega_i}\Phi_i, \\ W(\lambda^{\omega_i}\Phi_i) &= \lambda W(\Phi_i). \end{aligned}\right\} \tag{42}$$

In this case, one says that W is quasi-homogeneous with weights ω_i. The fixed point describes a conformally invariant model in which the conformal dimensions of Φ_i are $(h_i, \bar{h}_i) = (\frac{1}{2}\omega_i, \frac{1}{2}\omega_i)$. The main result is that a quasi-homogeneous function W uniquely characterizes an $N=2$ superconformal model up to field redefinitions (with a finite non-zero jacobian) and the addition of trivial quadratic terms in new fields. The analysis requires that Φ^n have a dimension n times that of Φ, which is valid as a consequence of the $N=2$ algebra. By studying the scaling of the partition function, and comparing with the Weyl anomaly formula (Polyakov 1981), one can show that the central charge is $c = 6\beta$, where

$$\beta = \sum_i (\tfrac{1}{2} - \omega_i) \tag{43}$$

is called the 'singularity index' of W.

An important notion in singularity theory is 'modality'. Roughly speaking, this is the number of parameters that characterize the model. It is not quite the same thing as the number of physical moduli. Remarkably, the classification of modality $m=0$ singularities precisely corresponds to the $N=2$ minimal models! The same ADE classification discussed earlier was

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known previously in singularity theory, because there is a prescription for associating Dynkin-like diagrams to singularity types. The $m = 0$ classification is

$$\left.\begin{array}{lll} \mathrm{A}_n: & x^{n+1} & k = n-1, \\ \mathrm{D}_n: & x^{n-1}+xy^2 & k = 2n-4, \\ \mathrm{E}_6: & x^3+y^4 & k = 10, \\ \mathrm{E}_7: & x^3+xy^3 & k = 16, \\ \mathrm{E}_8: & x^3+y^5 & k = 28. \end{array}\right\} \tag{44}$$

An obvious problem is to find the superpotentials that correspond to the various Kazama–Suzuki models.

The power of this approach is illustrated by the fact that there are three reducible minimal models, which are readily identified. Namely, as $\mathrm{A}_m \otimes \mathrm{A}_n$ corresponds to $x^{m+1}+y^{n+1}$,

$$\left.\begin{array}{l} \mathrm{D}_4 \equiv \mathrm{A}_2 \otimes \mathrm{A}_2, \\ \mathrm{E}_6 \equiv \mathrm{A}_2 \otimes \mathrm{A}_3, \\ \mathrm{E}_8 \equiv \mathrm{A}_2 \otimes \mathrm{A}_4. \end{array}\right\} \tag{45}$$

The first of the correspondences is proved by noting that a linear change of variables allows x^3+xy^2 to be expressed as the sum of two cubes. The discrete symmetries of the minimal models are also readily understood. For example, the Z_{k+2} symmetry of the A_{k+1} model is generated by $x \rightarrow \exp{[2\pi \mathrm{i}/(k+2)]}\, x$.

Each of Gepner's models is characterized by a superpotential given by the appropriate sum of polynomials. For example, the 3^5 model corresponds to five A_4 models: $W = \sum_{i=1}^{5} \Phi_i^5$. The corresponding Calabi–Yau space is given by the hypersurface $W = 0$ in CP^4. This fact is derived (Greene *et al.* 1988) by making an appropriate change of variables in the path integral

$$\int \mathrm{d}\Phi_1 \ldots \mathrm{d}\Phi_5 \exp\left\{\mathrm{i}\int \mathrm{d}^2x\, \mathrm{d}^2\theta\, W(\Phi_i)\right\}. \tag{46}$$

Only some of Gepner's models correspond to the fully classified complete-intersection Calabi–Yau (CICY) spaces given by polynomial constraints in products of complex projective spaces (Candelas *et al.* 1988*a*, *b*). The appropriate generalization that accommodates all of Gepner's models, suggested by the structure of quasi-homogeneous functions, involves 'weighted projective spaces'. The space $WCP^N_{k_1 \ldots k_{N+1}}$ is defined by the identification $[z_1, \ldots, z_{N+1}] \sim [\lambda^{k_1}z_1, \ldots, \lambda^{k_{N+1}}z_{N+1}]$. One subtlety is that these have non-trivial fixed-point sets. Remarkably, the condition for the vanishing of the first Chern class is simply that $c = 9$.

Concluding remarks

It was argued several years ago that string theory is a 'strongly coupled' theory (Dine & Seiberg 1985; Kaplanovsky 1985), meaning that (non-perturbative) quantum effects would be important. This could cause one to worry that the study of classical ground states is a waste of time, but I think that would be an overreaction. The three-generation models clearly come quite close to giving the desired phenomenology; it would be foolish not to explore how far they can be pushed. Many realistic features emerge quite naturally: gravity, popular gauge groups and representations, chiral families of fermions, axions, symmetry-breaking mechanisms, supersymmetry, etc. (I do not understand how those who say 'There is not a shred of experimental evidence for string theory' can ignore all these successes.) It would be a very strange coincidence if classical solutions were completely off the mark.

Certainly, non-perturbative phenomena will be crucial for a complete understanding. It is quite clear that the $N = 1, D = 4$ supersymmetry of the Calabi–Yau or orbifold solutions is not broken at any order in perturbation theory. Also, the dilaton and other massless states do not acquire a mass. I think it is reasonable to expect these things to happen in the complete non-perturbative quantum theory, however. An encouraging result is the recent demonstration that the string perturbation expansion diverges (Gross & Periwal 1988). This suggests that not every classical solution need correspond to a quantum ground state, but that whenever one does the needed symmetry breaking and mass generation could occur. Also, there probably are instantons that give quantum tunnelling between different classical vacua. These types of effects might lift the enormous degeneracy with which we are currently faced.

One question that has received much attention in recent years is why the cosmological constant is so small (less than 10^{-120} in Planck units). I don't know the answer, but let me offer the following comments. Perhaps, as we have discussed, the correct string theory ground state is reasonably approximated by a classical solution with $N = 1$ supersymmetry in four dimensions. Because supersymmetry is unbroken at every order in the loop expansion, the cosmological constant undoubtedly vanishes at every order. The mystery that then needs to be understood is why the non-perturbative effects that break supersymmetry do not generate a cosmological constant at the same time. It seems to me that the problem must be addressed in the context of the complete theory and is very unlikely to be resolved by considerations that are not sensitive to Planck-scale physics. I have assumed that low-energy supersymmetry is required to solve the 'hierarchy problem'. This was the principle motivation for looking for supersymmetric solutions, although they do seem to fit in rather naturally. Still, it would be very reassuring and helpful to have supporting experimental evidence. Could a 'string miracle' other than space-time supersymmetry do the job? After all, we need one to eliminate the cosmological constant.

In conclusion, it would probably be foolhardy to predict dramatic phenomenological successes for string theory in the near term. Still, there are some encouraging possibilities that deserve to be pursued. We might get lucky!

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Discussion

J. R. Ellis, F.R.S. (*Theory Division, CERN, Geneva, Switzerland*). Professor Schwarz has discussed extensively (2,2) compactifications. As he knows, more general compactifications are compatible with $N = 1$ space-time supersymmetry. Have any advances been made recently in the classification of such (2,1) and (2,0) compactifications?

J. H. Schwarz. As Dr Ellis probably knows, (2,0) and (2,1) models have been discussed in recent papers by Dine & Seiberg (1985), Distler & Greene (1988), and Cvetic. I am not an expert in these matters, but I am not aware of any recent advances indicating whether such schemes could be realistic.

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